Two geo-arithmetic representations of n^3: sum of hex numbers

Husan Unal
Yildiz Technical University, Turkey
<huu1932@fsu.edu>

Mathematics students in sixth-century BC Greece concentrated on four very separate areas of mathematics (called *mathemata*): *arithmetica* (arithmetic); *harmonia* (music); *geometria* (geometry); and, *astrologia* (astronomy). "This fourfold division of knowledge became known in the Middle Ages as the 'quadrivium'" (Burton, 1997, p. 88). To these early Greeks, arithmetic and geometry were as separate as music and astronomy. Mathematicians soon realised that arithmetic and geometry were not separate, and that some intriguing mathematics lay at their intersection. When we construct and sum series, there are structural parallels between the algebraic and geometrical representations.

Studies (e.g., Vinner, 1989; Tall, 1991) have shown that students' understanding is typically analytic and not visual. Two possible reasons for this are when the analytic mode, instead of the graphic mode, is most frequently used in instruction or, when students or teachers hold the belief that mathematics consists simply of skillful manipulation of symbols and numbers.

The National Council of Teachers of Mathematics (NCTM) states that: "Different representations support different ways of thinking about and manipulating mathematical objects. An object can be better understood when viewed through multiple lenses" (2000, p. 360).

This article presents two ways of visualising a series in *proof without words* (PWW) style (Nelsen, 1993; 2000). The contention is not that one representation is superior to another, only that students often construct vastly different personal and idiosyncratic representations which lead to different understandings of a concept. In his introduction, Nelsen (2000) states, "In my first introduction to the first collection of PWWs with their students… Respondents commented on using PWWs with classes at all levels — precalculus courses in high school, college courses in calculus, number theory, and combinatorics, and preservice and inservice classes for teachers" (p. x). Alsina and Nelsen's *Math Made Visual* (2006) raises an important question on the back cover of the book: "Is it possible to make mathematical drawings that help to understand mathematical ideas, proofs and arguments? The authors of this book are convinced that the answer is yes and the objective of this book is to show how some visualization techniques may be employed to produce pictures that have both mathematical and pedagogical interest."

PWWs have been used both in pre-service and inservice mathematics courses. Pandiscio (2001) discusses the importance of a single problem and points out: "[W]e must consider the type of problem we use. I urge teachers to think broadly and creatively when designing tasks, so that students learn as much as possible from their engagements with those task" (p. 100).

Proof without words

$$\sum_{k=1}^{n} 3k^2 - 3k + 1 = 1 + 7 + 19 + \ldots + \left(3n^2 - 3n + 1\right) = n^3$$

If we were to find this sum in figurative form, we need to find the general term that represents the algebraic form, which is $3n^2 - 3n + 1$. Figure 1 represents the general term: area of base rectangle plus area of stair-like shape (tower) gives us the nth term.

nth term:
$n(2n - 1) + (n - 1)^2$
$= 2n^2 - n + n^2 - 2n + 1$
$= 3n^2 - 3n + 1$

In Figure 2, the stair-like shape (tower) is restructured into a square having side $n - 1$.

Based on the general term, each element of the series is constructed, as shown in Figure 3.

In Figure 4, elements of the series were stacked together and it creates a tower.

Let us separate the total sum into two stairs, shown in Figure 5.

We have combined the two stairs in Figure 6, and it does create a cube with a dimension $n \times n \times n$, so total sum is equal to n^3.

Is there another way to solve this problem visually? The key is the finding of the general term in figurative form.

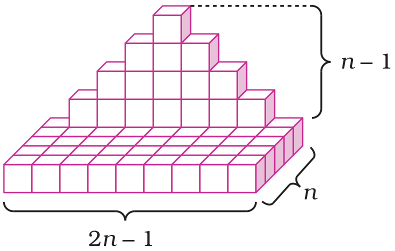

Figure 1. General term.

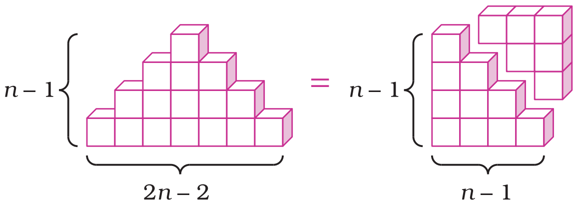

Figure 2. Restructuring.

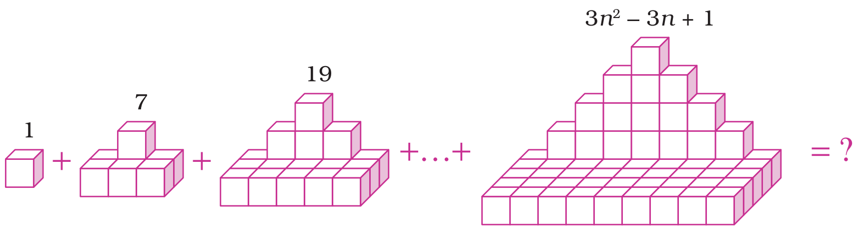

Figure 3. Elements of the series.

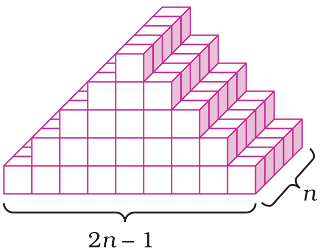

Figure 4. Total sum.

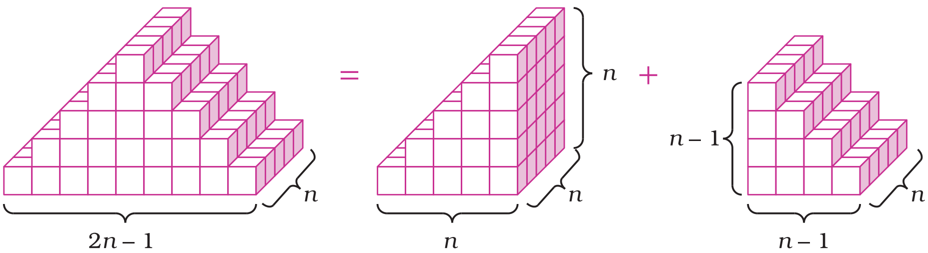

Figure 5. Separation.

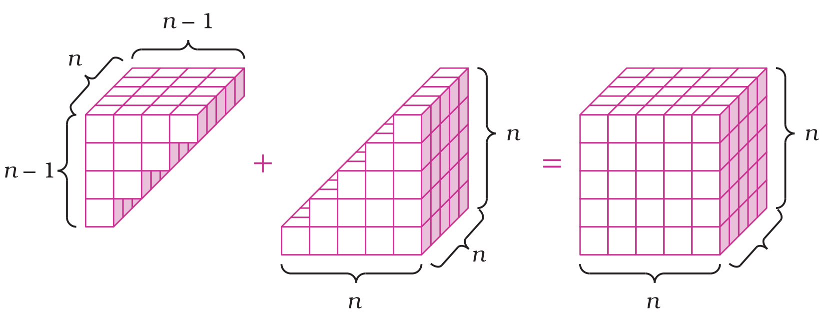

Figure 6.Re-structuring.

$$\sum_{k=1}^{n} 3k^2 - 3k + 1 = 1 + 7 + 19 + \ldots + \left(3n^2 - 3n + 1\right) = n^3$$

Figure 7 represents the general term. It consists of two rectangles: the first rectangle on the left hand side has the dimensions $1 \times (n-1)^2$ and the rectangle on the right hand side $n(2n-1)$.

Total area gives the general term:
$n(2n-1) + (n-1)^2$
$= 2n^2 - n + n^2 - 2n + 1 = 3n^2 - 3n + 1$

Figure 8 represents the elements of the series.

If we stack all the elements together, as in Figure 9, they create a rectangle with dimensions $n - n^2$, and the area gives the total sum, which is equal to n^3.

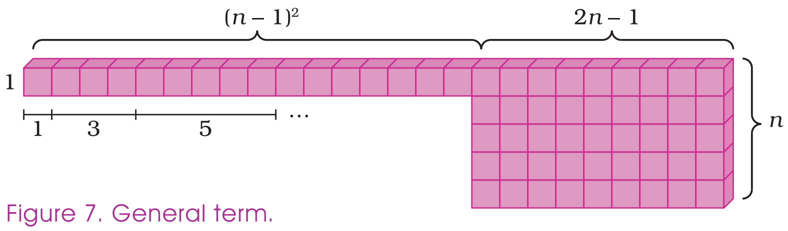

Figure 7. General term.

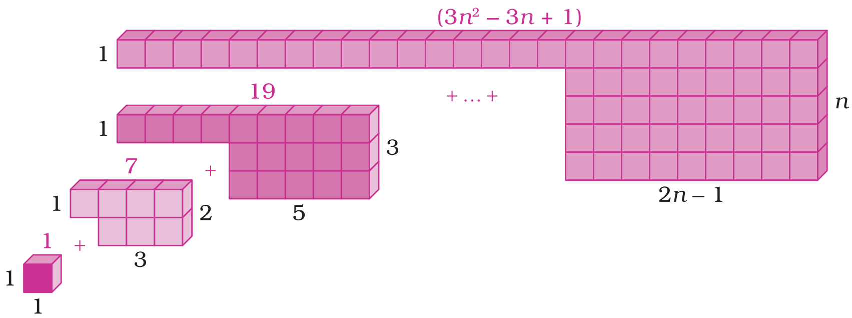

Figure 8. Elements of the series.

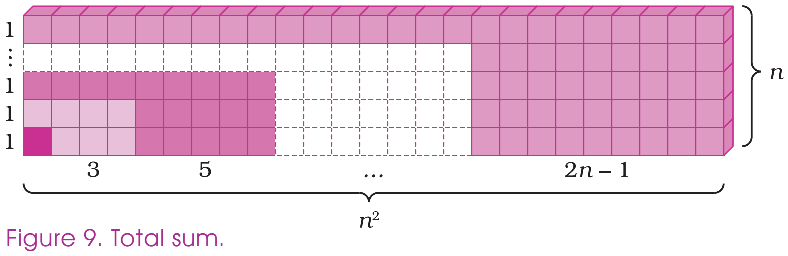

Figure 9. Total sum.

Krutetskii (1976) identified three main types of mathematical processing by learners: analytic, geometric, and harmonic. Analytic learners rely more on verbal-logical processing. On the other hand, geometric learners rely strongly on visual-pictorial processing. Harmonic learners rely on both visual and verbal processing.

The intention is not to support one way of processing over another. However, students do develop mathematical power by learning to recognise an idea embedded in a variety of different representational systems and by then translating the idea from one mode of representation to another.

Determining the viability of any method is an important step in mathematics teaching and learning. Often students believe there is a single method for solving a problem and that is that taught by the teacher. By reasoning through multiple methods students begin to think "outside the box." As Pandiscio (2001) points out: "What better mark of learning than to have a student present solution that we haven't seen before?" (p. 103).

References

Alsina, C. & Nelsen, B. R. (2006). *Math made visual: creating images for understanding mathematics*. Washington DC: Mathematical Association of America.

Burton, D. (1997). *The history of mathematics: An introduction*. New York: McGraw-Hill.

Janvier, C. (1987). Translation processes in mathematics education. In C. Janvier (ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 27–32). Hillsdale, NJ: Lawrence Erlbaum.

Krutetskii, V. (1976). *The psychology of mathematical abilities in school children*. Chicago: The University of Chicago Press.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Nelsen, B. R. (1993). *Proofs without words: Exercises in visual thinking*. Washington DC: Mathematical Association of America.

Nelsen, B. R. (2000). *Proofs without words: More exercises in visual thinking*. Washington DC: Mathematical Association of America.

Pandiscio E. (2001). Problem solving in middle-level geometry. *The Clearing House, 75*(2), 99–103.

Tall, D. (1991). Intuition and rigor: The role of visualization in the calculus. In W. Zimmermann & S. Cunningham (Eds), *Visualization in teaching and learning mathematics* (pp. 105–119). Washington DC: Mathematical Association of America.

Vinner, S. (1989). The avoidance of visual considerations in calculus students. *Focus on Learning Problems in Mathematics, 11*(2), 149–156.